to the amount of circumferential deformation for thermal expansion of the semifinished product. The correctness of this assumption follows from analyzing the solution in [9] if the considerable anisotropy of semifinished product material properties is considered. Extreme values of residual deflections of the inner surface obtained by experiment in [9] are shown by circles in Fig. 4. Straight line 2 relates to values of residual deflections for the central surface of the cylinder which is expressed in terms of residual deflection of front surfaces by the equation $w_* = 0.5\ (w(r_1) + w(r_2))$.

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DISLOCATIONS AND DISCLINATIONS IN NONLINEAR ELASTIC BODIES WITH MOMENT STRESSES

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A theory of dislocations and disclinations in elastic media which exhibit moment stresses and experience considerable strains is constructed. The marked effect of Volterra type dislocations in a Kosser nonlinearly elastic continuum is demonstrated by solving the problem of determining displacement and rotation fields in a multiconnected region with prescribed fields for the strain tensor and the bending strain tensor. Expression of Volterra dislocation characteristics in terms of the strain tensor field is given by means of a multiplicative contour integral. As a special case consideration is given to plane strain with which it is possible to delineate dislocations and disclinations in terms of normal contour integrals. Within the limits of moment nonlinear elasticity theory accurate solutions are found for the problem of screw dislocations and wedge dislocations. The effect of considering moment stresses and nonlinearity on the behavior of solutions close to the axis of a defect is analyzed.

1. In a model of a Kosser continuum [1-4] each particle of a solid has the degrees of freedom of an absolutely solid body. The position of particles in a deformed configuration is determined by radius-vector $\mathbf{R}$ and by strictly orthogonal tensor $\mathbf{H}$ called below the microrotation tensor. By using the principle of material indifference [5] it is possible to show that specific (per unit volume of reference configuration) potential energy W of an elastic Kosser continuum will depend on deformation of the body by means of two second rank tensors: tensor $\mathbf{U} = (\nabla^0\mathbf{R})\cdot\mathbf{H}^T$, called in the future the first measure of strain,

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and a tensor (more accurately pseudotensor) $\mathbf{L}$ called the first tensor of bending strain and determined from the relationship

$$\mathbf{L} \times \mathbf{E} = -(\nabla^0 \mathbf{H}) \cdot \mathbf{H}^T. \tag{1.1}$$

Here $\mathbf{E}$ is the unit tensor; $\nabla^0 = \mathbf{r}^s \partial/\partial x^s$ is nabla-operator of reference (undeformed) configuration; x^s are Lagrange coordinates. Vectors $\mathbf{r}^s$ are found from the equations $\mathbf{r}^s \cdot \mathbf{r}_k = \delta_k{}^s$, $\mathbf{r}_k = \partial \mathbf{r}/\partial x^k$ ($\delta_k{}^s$ is Kronecker symbol; $\mathbf{r}$ is radius-vector of a particle in reference configuration).

For a gyrotropic medium W will be a gyrotropic function of $\mathbf{U}$ and $\mathbf{L}$, i.e., the following requirement is satisfied

$$W(\mathbf{Q}^T \cdot \mathbf{U} \cdot \mathbf{Q},\ \mathbf{Q}^T \cdot \mathbf{L} \cdot \mathbf{Q}) = W(\mathbf{U},\ \mathbf{L}) \tag{1.2}$$

($\mathbf{Q}$ is any strictly orthogonal tensor).

By using a representation of strictly orthogonal tensor $\mathbf{H}$ in terms of finite rotation vector $\boldsymbol{\theta}$ [6]

$$\mathbf{H} = \mathbf{P}_+^{-1} \cdot \mathbf{P}_- = \mathbf{P}_- \cdot \mathbf{P}_+^{-1}, \quad \mathbf{P}_\pm = \mathbf{E} \pm \frac{1}{2} \mathbf{E} \times \boldsymbol{\theta}, \tag{1.3}$$

by means of (1.1) we obtain

$$\mathbf{L} = \frac{4}{4+\theta^2} (\nabla^0 \boldsymbol{\theta}) \cdot \left(\mathbf{E} + \frac{1}{2} \mathbf{E} \times \boldsymbol{\theta}\right), \quad \theta^2 = \boldsymbol{\theta} \cdot \boldsymbol{\theta}. \tag{1.4}$$

With linearization for the case of small strains of tensors $\mathbf{U}$ and $\mathbf{L}$ with respect to $\boldsymbol{\theta}$ and $\nabla^0 \mathbf{w}$ ($\mathbf{w} = \mathbf{R} - \mathbf{r}$ is displacement vector) we arrive at strain tensor e and a bending-twisting vector $\nabla^0 \boldsymbol{\theta}$, used [1] in linear moment theory of elasticity:

$$\mathbf{U} \approx \mathbf{E} + \mathbf{e}, \ \mathbf{e} = \nabla^0 \mathbf{w} + \mathbf{E} \times \boldsymbol{\theta}, \ \mathbf{L} \approx \nabla^0 \boldsymbol{\theta}. \tag{1.5}$$

For simplicity we assume that both mass external forces and loads, and loads distributed over the body surface are absent. Then from the variation principle $\delta \int\limits_v W dv = 0$ we obtain an equilibrium equation

$$\nabla^0 \cdot (\mathbf{T}^* \cdot \mathbf{H}) = 0, \ \nabla^0 \cdot (\mathbf{M}^* \cdot \mathbf{H}) + (\nabla^0 \mathbf{R}^T \cdot \mathbf{T}^* \cdot \mathbf{H})_\times = 0, \tag{1.6}$$
$$\mathbf{T}^* = \partial W/\partial \mathbf{U}, \ \mathbf{M}^* = \partial W/\partial \mathbf{L},$$

and boundary conditions

$$\mathbf{n} \cdot \mathbf{T}^* \cdot \mathbf{H} = 0, \ \mathbf{n} \cdot \mathbf{M}^* \cdot \mathbf{H} = 0 \text{ for } \partial v, \tag{1.7}$$

where v is the volume occupied by the elastic Kosser medium in reference configuration; $\mathbf{n}$ is normal to the boundary of the body ∂v; symbol $\mathbf{P}_\times$ vector invariant of second rank tensor $\mathbf{P}$: $(P^{ks}\mathbf{r}_k\mathbf{r}_s)_\times = P^{ks}\mathbf{r}_k \times \mathbf{r}_s$. By using the Piola identity [5] Eq. (1.6) may be written in spatial (Euler) coordinates X^α (α = 1, 2, 3):

$$\nabla \cdot \mathbf{T} = 0, \ \nabla \cdot \mathbf{M} + \mathbf{T}_\times = 0 \text{ in } V, \ \mathbf{N} \cdot \mathbf{T} = 0, \ \mathbf{N} \cdot \mathbf{M} = 0 \text{ on } \partial V; \tag{1.8}$$
$$\mathbf{T} = J^{-1} (\nabla^0 \mathbf{R})^T \cdot \mathbf{T}^* \cdot \mathbf{H}, \ \mathbf{M} = J^{-1} (\nabla^0 \mathbf{R})^T \cdot \mathbf{M}^* \cdot \mathbf{H}, \tag{1.9}$$
$$\nabla = \mathbf{R}^\alpha \partial/\partial X^\alpha, \ \mathbf{R}_\beta = \partial \mathbf{R}/\partial X^\beta, \ \mathbf{R}^\alpha \cdot \mathbf{R}_\beta = \delta_\beta{}^\alpha, \ J = \det(\nabla^0 \mathbf{R}).$$

Here ∇ is spatial gradient operator; V is the region occupied by the body in a deformed state; $\mathbf{N}$ is normal to ∂V; $\mathbf{T}$ is stress tensor; $\mathbf{M}$ is moment stress tensor.

The equations of nonlinear moment theory of elasticity formed have been obtained by other methods [2-4]. It is noted that models of a solid taking account of moment stresses are used in describing structurally-inhomogeneous media [7], in liquid crystal theory [8], and in other questions of deformed body mechanics.

Tensors $\mathbf{U}$ and $\mathbf{L}$ are similar to the Cauchy–Green measure of strain [5]. By changing the places of reference and deformed configurations, i.e., by making the changes $\mathbf{R} \to \mathbf{r}$,

$\nabla^0 \to \nabla, \theta \to -\theta$, we obtain tensors which are analogs of the measure of Almanzi strain [5] in classical nonlinear elasticity theory:

$$\mathbf{u} = \nabla \mathbf{r} \cdot \mathbf{H}, \quad \mathbf{l} = -\frac{4}{4+\theta^2}(\nabla\theta) \cdot \left(\mathbf{E} - \frac{1}{2}\mathbf{E} \times \theta\right). \tag{1.10}$$

Tensors $\mathbf{u}$ and $\mathbf{l}$ will be called the second measure of strain and the second tensor of strain respectively. From (1.3), (1.4), (1.10) the following relationship emerges

$$\mathbf{u}^{-1} = \mathbf{H}^T \cdot \mathbf{U} \cdot \mathbf{H}, \ \mathbf{u}^{-1} \cdot \mathbf{l} = -\mathbf{H}^T \cdot \mathbf{L} \cdot \mathbf{H}. \tag{1.11}$$

By placing $\mathbf{Q} = \mathbf{H}$ in (1.2) on the basis of (1.11) we obtain for a gyrotropic material $W = W(\mathbf{u}, \mathbf{l})$. In exactly the same way it is possible to show that in a gyrotropic Kosser continuum stress tensors $\mathbf{T}$ and moment stresses $\mathbf{M}$ on strain of the medium through the second measure of strain and the second bending strain tensor.

Definitive relationships for materials with bonds are constructed by means of introducing Lagrange multipliers [5]. Imposition of a condition of incompressibility

$$\det \mathbf{U} = 1 \tag{1.12}$$

leads to an addition in expression (1.9) for $\mathbf{T}$ independent of strain of spherical tensor $p\mathbf{E}$. Identification of microrotation tensor $\mathbf{H}$ with macrorotation tensor $\mathbf{A} = (\nabla^0\mathbf{R} \cdot \nabla^0\mathbf{R}^T)^{-1/2} \cdot \nabla^0\mathbf{R}$ leads to vector connection

$$\mathbf{U}_{\times} = 0. \tag{1.13}$$

Linearization of (1.13) taking account of (1.5) gives a known relationship for a Kosser continuum $\theta = (1/2)\nabla^0 \times \mathbf{w}$ [9]. In this case the expression for $\mathbf{M}$ does not change and the expression for tensor $\mathbf{T}$ for incompressible materials takes the form

$$\mathbf{T} = p\mathbf{E} + (\nabla^0\mathbf{R})^T \cdot (\partial W/\partial \mathbf{U} + \mathbf{q} \cdot \mathbf{D}) \cdot \mathbf{H}, \tag{1.14}$$

where $\mathbf{D} = -\mathbf{E} \times \mathbf{E}$ is Levy–Civita tensor [5, 6]; $\mathbf{q}$ is independent of strain vector.

2. We consider the problem of determining displacement and microrotation fields for a Kosser continuum from known fields for tensor $\mathbf{u}$ and $\mathbf{l}$, which are prescribed as twice differentiated functions of Euler coordinates X^α. From (1.3), (1.10) we have

$$\partial \mathbf{H}^T/\partial X^\alpha = \mathbf{\Pi}_\alpha \cdot \mathbf{H}^T, \ \mathbf{\Pi}_\alpha = -\mathbf{E} \times (\mathbf{R}_\alpha \cdot \mathbf{l}). \tag{2.1}$$

Necessary and sufficient conditions for resolution of these equations with respect to $\mathbf{H}$ contain nine independent relationships and they have the form

$$\partial \mathbf{\Pi}_\beta/\partial X^\alpha - \partial \mathbf{\Pi}_\alpha/\partial X^\beta = \mathbf{\Pi}_\alpha \cdot \mathbf{\Pi}_\beta - \mathbf{\Pi}_\beta \cdot \mathbf{\Pi}_\alpha. \tag{2.2}$$

As in [10], the solution of Eqs. (2.1) may be written by means of a multiplicative integral [11]

$$\mathbf{H}\ (M) = \overset{\frown}{\int_{M_0}^{M}} (\mathbf{E} + d\mathbf{R} \cdot \mathbf{\Pi}) \cdot \mathbf{H}_0^T, \quad \mathbf{\Pi} = \mathbf{R}^\alpha \mathbf{\Pi}_\alpha. \tag{2.3}$$

Here M_0 is a point of region V in which the initial value of tensor $\mathbf{H}(M_0) = \mathbf{H}_0$; is prescribed; M is current point. In singly connected region V the value of $\mathbf{H}(M)$ does not depend on choice of the curve connecting points M_0 and M. After determining $\mathbf{H}$ by Eq. (2.3) the position of particles of the medium in the reference configuration is found from (1.10) in quadratures

$$\mathbf{r}(M) = \int_{M_0}^{M} d\mathbf{R} \cdot (\mathbf{u} \cdot \mathbf{H}^T) + \mathbf{r}(M_0). \tag{2.4}$$

Necessary and sufficient conditions for independence of the integral in (2.4) from the integration path in a singly connected region consist of fulfilling the equalities

$$\mathbf{R}^\alpha \times (\partial \mathbf{u}/\partial X^\alpha) + \mathbf{R}^\alpha \times \mathbf{u} \cdot \mathbf{\Pi}_\alpha = 0. \tag{2.5}$$

Conditions (2.2) and (2.5) which consist of 18 scalar relationships are relationships for compatability of strains in nonlinear moment elasticity theory. Similar relationships for deformation tensors $\mathbf{U}-\mathbf{E}$ and $\mathbf{L}$, prescribed as functions of Lagrange coordinates were obtained in [3].

If region V occupied by an elastic body in a deformed condition is multiconnected, then displacements $\mathbf{w}=\mathbf{R}-\mathbf{r}$ and microrotations determined by Eqs. (2.3) and (2.4) will generally speaking be single-valued. The single value nature is eliminated if the region is transformed into a singly connected region with provision of the required number of sections. Vectors $\mathbf{r}$ and $\boldsymbol{\theta}$ may undergo jumps on intersecting each section. It is possible to show by the method in [10] that jumps are described by the relationships

$$\mathbf{H}_+=\boldsymbol{\Omega}\cdot\mathbf{H}_-,\quad \boldsymbol{\theta}_+=\frac{4}{4-\boldsymbol{\omega}\cdot\boldsymbol{\theta}_-}\left(\boldsymbol{\omega}+\boldsymbol{\theta}_-+\frac{1}{2}\,\boldsymbol{\theta}_-\times\boldsymbol{\omega}\right), \tag{2.6}$$
$$\boldsymbol{\omega}=2(1+\operatorname{tr}\boldsymbol{\Omega})^{-1}\boldsymbol{\Omega}_\times,\ \mathbf{r}_+=\boldsymbol{\Omega}\cdot\mathbf{r}_-+\mathbf{b},$$

where $\boldsymbol{\Omega}$ is a strictly orthogonal tensor constant for all points of each section; $\boldsymbol{\omega}$ and $\mathbf{b}$ are constant vectors. Equations (2.6) mean that if a nonlinearly elastic Kosser body occupying a multiconnected region in a stressed state in which measures of Almazi type strain $\mathbf{u}$, $\mathbf{l}$ (and consequently stress tensor $\mathbf{T}$ and moment stress tensor $\mathbf{M}$) is cut, then in an unstressed state the position of opposite edges of a section will differ from each other in rigid movement. A similar proof for a nonlinearly elastic medium in moment stresses is given in [10].

In the case of doubly connected region $\boldsymbol{\Omega}$, $\mathbf{b}$ are expressed in terms of a field for strain tensors $\mathbf{u}$, $\mathbf{l}$ by equations similar to those provided in [10]:

$$\boldsymbol{\Omega}^T=\mathbf{H}_0\cdot\widehat{\oint}(\mathbf{E}+d\mathbf{R}\cdot\boldsymbol{\Pi})\cdot\mathbf{H}_0^T, \tag{2.7}$$
$$\mathbf{b}=\oint d\mathbf{R}'\cdot\mathbf{u}(\mathbf{R}')\cdot\widehat{\int_{M_0}^{M'}}(\mathbf{E}+d\mathbf{R}\cdot\boldsymbol{\Pi})\cdot\mathbf{H}_0^T+\mathbf{r}_0\cdot(\mathbf{E}-\boldsymbol{\Omega}^T).$$

Thus, it is shown that in nonlinearly elastic bodies with moment stresses defects in the form of Volterra dislocations may exist. As in [10], defect parameters $\mathbf{b}$ and $\boldsymbol{\omega}$ are called Burgers vector and Frank vector respectively. The set of equations for determining the stressed state for a nonlinearly elastic Kosser medium containing Volterra dislocations with prescribed characteristics $\mathbf{b}$ and $\boldsymbol{\omega}$ consist of equilibrium Eqs. (1.8) in which tensors $\mathbf{T}$ and $\mathbf{M}$ are expressed in terms of $\mathbf{u}$ and $\mathbf{l}$, compatibility Eqa. (2.1), (2.5), and relationships (2.7).

Similar to the previous occasion we consider the problem of determining displacements and microrotations in a non-singly connected region occupied by a Kosser medium in an undeformed configuration from known fields for tensors $\mathbf{U}$ and $\mathbf{L}$, prescribed as continuous and twice differentiated functions of Lagrange coordinates.

3. By limiting ourselves to the case of plane strain described by the relationships

$$X_1=X_1(x_1,x_2),\ X_2=X_2(x_1,x_2),\ X_3=x_3, \tag{3.1}$$

where x_k and X_k are coordinates of points for a medium on a Cartesian basis $\{\mathbf{e}_k\}$ before and after deformation respectively, it is possible to simplify statement of the problem for stresses created by an isolated defect, and in particular to obtain an expression for its characteristics in terms of normal contour integrals. We introduce complex coordinates [5, 12]

$$\zeta=x_1+ix_2,\ \bar{\zeta}=x_1-ix_2,\ z=X_1+iX_2,\ \bar{z}=X_1-iX_2.$$

Plane strain (3.1) is prescribed by means of a complex-sign function

$$z=z(\zeta,\bar{\zeta}),\ X_3=x_3. \tag{3.2}$$

By considering the case when the region occupied by a body in the undeformed condition is not singly connected we assume that tensors $\mathbf{U}$ and $\mathbf{L}$ are prescribed:

$$\mathbf{L}=L_1(\zeta,\bar{\zeta})\,\mathbf{f}^1\mathbf{f}_3+L_2(\zeta,\bar{\zeta})\mathbf{f}^2\mathbf{f}_3; \tag{3.3}$$
$$\mathbf{U}=U_\alpha{}^\beta(\zeta,\zeta)\,\mathbf{f}^\alpha\mathbf{f}_\beta+\mathbf{f}^3\mathbf{f}_3,\ \alpha,\beta=1,2. \tag{3.4}$$

Here $\{\mathbf{f}^\alpha\}$, $\{\mathbf{f}_\beta\}$ are complex bases corresponding to complex coordinates ζ, $\bar{\zeta}$ [12], $\mathbf{f}^3=\mathbf{f}_3=\mathbf{e}_3$.

We find tensor $\mathbf{H}$ in the form

$$\mathbf{H} = \exp(i\chi)\,\mathbf{f}^1\mathbf{f}_1 + \exp(-i\chi)\,\mathbf{f}^2\mathbf{f}_2 + \mathbf{f}^3\mathbf{f}_3. \tag{3.5}$$

In this general arrangement the rotation tensor with plane strain χ, which is subject to determination, is the angle of finite rotation for particles of the medium. By substituting (3.3), (3.5) in (1.1), we obtain

$$\partial\chi/\partial\zeta = L_1,\ \partial\chi/\partial\bar{\zeta} = L_2. \tag{3.6}$$

Resolution condition (3.6) with respect to χ is written as

$$\partial L_1/\partial\bar{\zeta} = \partial L_2/\partial\zeta. \tag{3.7}$$

By comparing the expression for the strain gradient $\nabla^0\mathbf{R}$, corresponding to transformation (3.2) with the expression $\nabla^0\mathbf{R} = \mathbf{U}\cdot\mathbf{H}$, obtained from determining $\mathbf{U}$ taking account of (3.4) and (3.5) we find that

$$\partial z/\partial\zeta = U_1^1\exp(i\chi),\ \ \partial z/\partial\bar{\zeta} = U_2^1\exp(i\chi).$$

According to (3.6) the resolution condition for these equations has the form

$$\partial U_2^1/\partial\zeta - \partial U_1^1/\partial\bar{\zeta} + i\,(L_1U_2^1 - L_2U_1^1) = 0. \tag{3.8}$$

Equations (3.7) and (3.8) are compatability equations with plane strain for the medium. They are equivalent to three real equations.

Analysis of the nature of multi-value for functions χ and z in a doubly connected region is carried out similar to [12]. In particular, in the section transforming the region into a single-value one the limiting values of functions χ and z are connected by the relationships

$$\chi_+ - \chi_- = K,\quad K = \oint L_1 d\zeta + L_2 d\bar{\zeta}; \tag{3.9}$$

$$z_+ = z_-\exp(iK) + \beta; \tag{3.10}$$

$$\beta = z_0\,(1 - \exp(iK)) + \oint \exp(i\chi)\left(U_1^1 d\zeta + U_2^1 d\bar{\zeta}\right) \tag{3.11}$$

(z_0 is a prescribed value of function z at a certain point of the region). Relationship (3.10) is a generalization of the Weingarten theory of linear elasticity theory in the case of plane strain. Its real form is as follows

$$\mathbf{w}_+ - \mathbf{w}_- = \frac{4}{4+\omega^2}\,\omega\times\left(\mathbf{R}_- + \frac{1}{2}\,\omega\times\mathbf{R}_-\right) + \mathbf{b}$$

($\omega = 2\tan(K/2)\,\mathbf{e}_3$ is Frank vector, $\mathbf{b} = \mathrm{Re}\beta_{\ 1} + \mathrm{Im}\,\beta\mathbf{e}_2$ is Buergers vector).

The boundary problem for plane strain of a body containing an isolated defect with prescribed characteristics consists of equilibrium Eqs. (1.6), boundary conditions (1.7) in which tensors $\mathbf{T}^*$ and $\mathbf{M}^*$ are expressed in terms of $\mathbf{U}$ and $\mathbf{L}$, compatability Eqs. (3.7) and (3.8), and relationships (3.9) and (3.11) prescribing dislocation parameters.

As an example we consider the problem of determining mechanical fields created by an isolated defect in an elastic ring $a \le r \le b$. Tensors $\mathbf{U}$ and $\mathbf{L}$, which satisfy the compatability conditions and equilibrium equations are found in the form

$$\mathbf{L} = L_r(r)\,\mathbf{e}_r\mathbf{e}_z + L_\varphi(r)\,\mathbf{e}_\varphi\mathbf{e}_z; \tag{3.12}$$

$$\mathbf{U} = U_1(r)\,\mathbf{e}_r\mathbf{e}_r + U_2(r)\,\mathbf{e}_\varphi\mathbf{e}_\varphi + \mathbf{e}_z\mathbf{e}_z, \tag{3.13}$$

where r, φ, z are cylindrical coordinates; $\mathbf{e}_r$, $\mathbf{e}_\varphi$, $\mathbf{e}_z$ are reference vectors corresponding to them. In this case Eq. (3.7) takes the form $dL_\varphi/dr + L_\varphi/r = 0$, whence according to (3.9) we obtain

$$L_\varphi(r) = K/(2\pi r). \tag{3.14}$$

Relationships (3.8) taking account of (3.12)-(3.14) are transformed:

$$L_r(r) \equiv 0, \quad dU_2/dr + U_2/r + \varkappa U_1/r = 0, \quad \varkappa = (2\pi + K)/2\pi. \tag{3.15}$$

Assuming that $\chi = 0$ with $\varphi = 0$ we find that $\chi = (\varkappa - 1)\varphi$. The microrotation tensor $\mathbf{H}$ is determined in the same way by (3.5). By calculating β from (3.11), similar to [12] it is possible to show that representation (3.12), (3.13) describes the deformed state of a cylinder with a wedge disclination.

A study of the stressed state is carried out for a "physically linear" material whose specific potential energy is taken in the form

$$W = \frac{\lambda}{2}\operatorname{tr}^2\varepsilon + \frac{\mu+\alpha}{2}\operatorname{tr}(\varepsilon\cdot\varepsilon^T) + \frac{\mu-\alpha}{2}\operatorname{tr}\varepsilon^2 + \tag{3.16}$$
$$+ \frac{\delta}{2}\operatorname{tr}^2\mathbf{L} + \frac{\gamma+\eta}{2}\operatorname{tr}(\mathbf{L}\cdot\mathbf{L}^T) + \frac{\gamma-\eta}{2}\operatorname{tr}\mathbf{L}^2, \quad \varepsilon = \mathbf{U} - \mathbf{E}$$

(λ, μ, α, δ, η, γ are elastic constants). Taking this into account it is possible to show that the equilibrium equations in moments are satisfied identically, and from the equilibrium equations in stresses one remains nontrivial

$$(\lambda + 2\mu)\frac{dU_1}{dr} + \lambda\frac{dU_2}{dr} + [\lambda(1-\varkappa) + 2\mu]\frac{U_1}{r} + [\lambda(1-\varkappa) - 2\mu\varkappa]\frac{U_2}{r} = 2\frac{(\lambda+\mu)(1-\varkappa)}{r}. \tag{3.17}$$

Boundary conditions (1.7) are reduced to the relationship

$$(\lambda + 2\mu)U_1(r) + \lambda U_2(r) = 2(\lambda + \mu), \; r = a, b. \tag{3.18}$$

Of special interest is the case of a solid cylinder ($a = 0$). By solving boundary problems (3.15), (3.17), (3.18) and directing parameter a to zero, for functions U_1 and U_2 we obtain

$$U_1 = \frac{1-2\nu}{1-\nu}\frac{\varkappa}{1+\varkappa}\rho^{\varkappa-1} + \frac{1}{(1+\varkappa)(1-\nu)}, \tag{3.19}$$
$$U_2 = \frac{1-2\nu}{1-\nu}\frac{\varkappa}{1+\varkappa}\rho^{\varkappa-1} + \frac{\varkappa}{(1+\varkappa)(1-\nu)}, \quad \rho = \frac{r}{b}, \quad \nu = \frac{\lambda}{2(\lambda+\mu)}.$$

Expressions for components of stress tensor $\mathbf{T}$ do not have singularity on the dislocation axis and they coincide with those calculated in momentless nonlinear elasticity theory [12].

The components of moment stress tensor $\mathbf{M}$ differing from zero are written in the form

$$M_{23} = (\gamma + \eta)\frac{\varkappa - 1}{rU_1(r)}, \quad M_{32} = (\gamma - \eta)\frac{\varkappa - 1}{rU_1(r)U_2(r)}. \tag{3.20}$$

From these relationships and (3.19) it can be seen that in a solid cylinder moment stress M_{23} has singularity of the order of ρ^{-1} with $\varkappa > 1$ and $\rho^{-\varkappa}$ with $\varkappa < 1$, and stress M_{32} has singularity of the order ρ^{-1} with $\varkappa > 1$ and $\rho^{1-2\varkappa}$ with $\varkappa < 1$. Linearization of expressions (3.20) with respect to parameter $\varkappa - 1$ is carried out with $\rho > 0$ to equations known from linear elasticity theory [13] according to which with $\rho \to 0$ moment stresses have a singularity of the order of ρ^{-1} for all $\varkappa \neq 1$.

On the basis of (3.16) taking account of (3.14) and (3.19) it is easy to establish that in calculating moment stresses the potential energy arriving in a unit length of a cylinder with a disclination and calculated by the equation

$$\Pi = g(\varkappa)\int_a^b rW(r)\,dr, \quad g(\varkappa) = \begin{cases} 2\pi, & \varkappa \leqslant 1, \\ 2\pi\varkappa^{-1}, & \varkappa > 1, \end{cases}$$

has a logarithmic singularity with $a \to 0$.

4. We consider the following transformation of reference configuration to current configuration:

$$R = R(r), \; \Phi = \varphi, \; Z = a\varphi + z \tag{4.1}$$

(r, φ, z, and R, Φ, Z are cylindrical coordinates in the reference and current configurations respectively). This transformation describes strain for a cylinder containing a screw dislocation with Burgers vector $\mathbf{b} = 2\pi a \mathbf{e}_z$. We prescribe the representation of microrotation tensor $\mathbf{H}$:

$$\mathbf{H} = \mathbf{e}_r\mathbf{e}_r + \cos\chi(r)(\mathbf{e}_\varphi\mathbf{e}_\varphi + \mathbf{e}_z\mathbf{e}_z) + \sin\chi(r)(\mathbf{e}_\varphi\mathbf{e}_z - \mathbf{e}_z\mathbf{e}_\varphi). \tag{4.2}$$

Measures of strain $\mathbf{U}$ and $\mathbf{L}$ corresponding to (4.1) and (4.2) have the form

$$\mathbf{U} = \frac{dR}{dr}\mathbf{e}_r\mathbf{e}_r + \frac{1}{r}(R\cos\chi + a\sin\chi)\mathbf{e}_\varphi\mathbf{e}_\varphi + \tag{4.3}$$
$$+ \frac{1}{r}(a\cos\chi - R\sin\chi)\mathbf{e}_\varphi\mathbf{e}_z + \sin\chi\mathbf{e}_z\mathbf{e}_\varphi + \cos\chi\mathbf{e}_z\mathbf{e}_z;$$
$$\mathbf{L} = \frac{d\chi}{dr}\mathbf{e}_r\mathbf{e}_r + \frac{\sin\chi}{r}\mathbf{e}_\varphi\mathbf{e}_\varphi + \frac{\cos\chi - 1}{r}\mathbf{e}_\varphi\mathbf{e}_z. \tag{4.4}$$

We limit ourselves to considering an uncompressible pseudocontinuum, i.e., we shall assume that connections (1.12) and (1.13) are applied to the material. This makes it possible to find straight away functions R(r) and $\chi(r)$:

$$R(r) = \sqrt{r^2 + A}, \quad \chi(r) = \operatorname{arctg}\frac{a}{r + \sqrt{r^2 + A}}.$$

Constant A is determined from the boundary conditions. In particular, for a solid cylinder R(0) = 0, consequently A = 0 and

$$R(r) = r, \ \chi(r) = \operatorname{arctg}(a/2r). \tag{4.5}$$

We consider a material with energy $W = 2\mu\ \mathrm{tr}\,\boldsymbol{\varepsilon} + \frac{\delta}{2}\mathrm{tr}^2\mathbf{L} + \frac{\gamma+\eta}{2}\mathrm{tr}(\mathbf{L}\cdot\mathbf{L}^T) + \frac{\gamma-\eta}{2}\mathrm{tr}\,\mathbf{L}^2$.

The problem of a screw dislocation for these materials without taking account of moment stresses was studied in [14]. In this case taking account of (1.14), (4.3)-(4.5) equilibrium (1.8) for a solid cylinder is reduced to four relationships

$$\frac{dp}{dr} = -\frac{2\mu}{r} + \frac{4\mu}{\sqrt{4r^2 + a^2}} + (\delta + 2\gamma)\frac{4a^2}{r(4r^2 + a^2)^2},$$
$$q_r = -(\delta + 2\gamma)\frac{4a}{(4r^2 + a^2)^{3/2}}, \quad q_\varphi = 0, \quad q_z = 0.$$

Here $q_r = \mathbf{q}\cdot\mathbf{e}_r$; $q_\varphi = \mathbf{q}\cdot\mathbf{e}_\varphi$; $q_z = \mathbf{q}\cdot\mathbf{e}_z$. As in [14], by means of these equations it is possible to show that in the vicinity of the dislocation axis tangential stress $T_{\varphi z}$ is directed towards a finite limiting value 2μ, and stress $T_{z\varphi}$ increases in proportion to r^{-1} as in linear elasticity theory. Moment stresses M_{rr} and $M_{\varphi\varphi}$ are written in the form

$$M_{rr} = a\left(\frac{\beta}{r\sqrt{4r^2 + a^2}} - \frac{2(\beta + 2\gamma)}{4r^2 + a^2}\right), \tag{4.6}$$
$$M_{\varphi\varphi} = a\left(\frac{2\beta}{4r^2 + a^2} - \frac{4\beta r}{(4r^2 + a^2)^{3/2}} + \frac{2\gamma}{r\sqrt{4r^2 + a^2}}\right).$$

If in solving (4.6) with r > 0 only first order terms with respect to parameter a are retained, then we arrive at solution of the problem of a screw dislocation within the limits of linear theory for a Kosser pseudocontinuum:

$$M^0_{rr} = -\gamma a/r^2, \quad M^0_{\varphi\varphi} = \gamma a/r^2. \tag{4.7}$$

According to (4.6) and (4.7) the solution of linear theory has with $r \to 0$ a stronger singularity in moment stresses compared with nonlinear theory. With an increase in distance from the dislocation axis the difference between the solutions of linear and nonlinear theory decrease, i.e., $M_{rr} \sim M_{rr}{}^0$, $M_{\varphi\varphi} \sim M_{\varphi\varphi}{}^0$ with $r \to \infty$. The potential energy arising in a unit length of a cylinder with a dislocation has a logarithmic singularity on the cylinder axis.

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USE OF THE SCATTERED LIGHT METHOD IN ORDER TO DETERMINE THE STRESS INTENSITY FACTOR K_{III} IN THREE-DIMENSIONAL PROBLEMS

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Strain "freezing" [1, 2] and scattered light [3] methods are used for experimental determination of the stress intensity factor (SIF) K_{III} in studying solid structural elements with surface or internal cracks. The scattered light method exhibits considerable potential and marked advantages over the "freezing" method by making it possible to obtain the required data without cutting up the model. However, this method has not been used extensively due to the complexity of experiments and interpretation of measured results. For example, in [3] it is suggested that the model is examined in a plane perpendicular to the crack front by a light beam intersecting the tip of the crack. This illumination scheme requires careful selection of the immersion liquid and treatment of the crack edge surfaces, and also rotation of the model or the device around the point of intersection of the crack front by the beam.

A simpler procedure is described in the present work for carrying out an experiment which makes it possible to carry over methods known in plane photoelasticity for treating experimental data for determining the SIF to the case of determining K_{III} for spatial cracks.

For longitudinal shear stresses close to the tip of a crack are expressed as follows:

$$\begin{gathered}\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0,\\ \tau_{xz} = K_{III}\,(2\pi r)^{-1/2}\sin(\theta/2), \quad \tau_{yz} = K_{III}(2\pi r)^{-1/2}\cos(\theta/2),\end{gathered} \tag{1}$$

where x, y, z is an orthogonal coordinate system orientated so that axis z is tangential to the crack front at point O (Fig. 1); r, θ are polar coordinates.

Since the value of optical difference for the path of the light beam due to the difference in quasiprincipal stresses which operate in a plane perpendicular to the illuminating beam is measured by the scattered light method, then it will be most effective to examine it in plane xOy parallel to axis x. It is important that the direction of principal stresses

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